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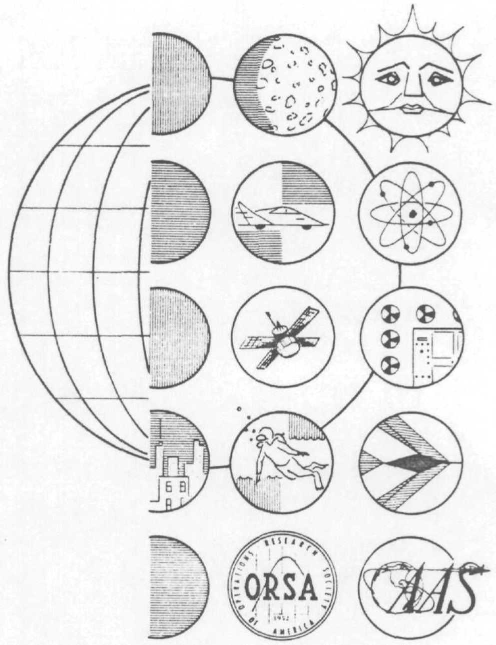

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ON THE ADJUSTMENT OF INDIRECT FIRES BY USE OF QUASILINEARIZATION

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QUASILINEARIZATION

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ABSTRACT

A method is presented for the adjustment of artillery fires even when the fall of shot cannot be observed, on the basis of data obtained in tracking the projectile over the first portion of its trajectory. The paper opens with a brief discussion of the physical situation and typical operational requirements. The equations of the projectile's motion are presented. A simple set of observations is assumed and a quasilinearization technique is developed which allows a determination of those trajectory parameters which yield the best least-squares fit to the observations. The paper is concluded by a consideration of some of the properties of the measurements which will actually be made by the radar. This leads to a consideration of certain statistical issues wherein it is suggested that some improvement in the character of the system parameter estimates could be effected by utilizing cost functionals other than the usual sum of the squared deviations. It is argued that utilization of one of these modified criterion functions permits one to account for the fact that the reliability of the empirically computed projectile velocity components varies in a partially known fashion along the projectile's trajectory. This allows us to give greater weight to those pieces of data in which we have the most confidence.

Warships are often called upon to fire against inland targets. In many cases only the geographic position (latitude, longitude, and altitude) of the target is known and observation of the fall of shot is not feasible. Certain variables affecting the projectile's flight will typically not be known precisely and this will cause uncertainty about the actual point of impact. It is clearly desirable to make efficient use of all data which might aid in the estimation of these variables. In this paper we discuss the use of quasilinearization methods in dealing with some of these estimation problems.

The case we shall deal with is that of a gun launched, fin stabilized, rocket boosted, unguided projectile. With rocket thrust set to zero this will also cover the case of unboosted, fin stabilized projectiles. (The very complex question of sabot separation for hypervelocity, unboosted projectiles will not, however, be treated here.) Rail or tube launched rocket projectiles may also be covered, simply by setting initial velocity to zero. Gun launched, spin stabilized projectiles may be dealt with by a slight extension of the present methods, accounting for gyroscopic drift.

It will be supposed that the ship is in coastal waters firing at a target some distance inland and below the ship's horizon. Without the aid of outside observers it is not possible to observe the fall of shot. The ship can, however, employ its fire control radar to track the shell for the first portion of its flight. The first projectile will be fired on the basis of estimates of the critical parameters. Tracking the projectile's flight, however, will permit the ship to correct its estimates. Tracking

With respect to such a system a projectile will be acted upon by the forces of gravity, drag, thrust, and lift, as well as a "Coriolis Force". The latter will be neglected, as it involves no unknown variables. Since the projectile can be observed over but a few miles of flight, we may as well assume that gravitational acceleration is constant and in the negative y direction. We visualize the projectile as a body of rotation, stabilized so as to maintain zero angle of attack. Thus, there is no lift force.

Generally, the projectile's booster motor will be designed to operate at some constant thrust. Thrust will be terminated at some predetermined time, with the projectile coasting thereafter. More complex thrust histories are easily treated, provided the form of the thrust-vs.-time function is known.

There is also the question of the wind's effect upon the projectile's flight. It is convenient to resolve the wind into in-range (x direction) and cross-range (z direction) components. Since the projectile is trimmed for zero angle of attack, the effect of cross winds is simply to induce cross-range drift at the wind's velocity. Since a projectile might remain in flight for anything up to 100 seconds, a 25-knot (14 yards/second) crosswind could cause as much as 1400 yards drift. Since drag forces are with respect to the air mass rather than the earth, the velocity term in the drag equation must reflect the in-range wind.

In general, of course, the wind will not be constant over the projectile's flight. In principle one could estimate the wind as a function of position along the flight path, given sufficient data. Since we, however, are flight of big-gun projectiles. Since angle of attack is constant in our case, C_D is a function only of Mach number. And since Mach number depends on velocity and density, we may write

$$C_D = C_D(v, \rho) \tag{2}$$

We will assume that an experimental determination of C_D has already been made.

The density of the atmosphere decreases with altitude. Its dependence on altitude is usually representable in the form

$$\rho(y) = \rho(0)e^{-hy} \tag{3}$$

where h is a suitable dimensioned constant. We shall assume that equation (3) holds but that h is not known precisely. The sealevel density, $\rho(0)$, may be easily and accurately measured. We write $\rho(0) = \rho_o$.

We define

$$U(t) = \begin{cases} 0 \text{ for } t<0 \\ 1 \text{ for } t\geq 0 \end{cases} \tag{4}$$

Assuming, for convenience, that $S_x=S_z=0$, we can now write down the projectile's equations of motion:

$$\left.\begin{aligned} m\ddot{x} &= [(1-U(t-t_c))T-D] \cos \Theta \\ m\ddot{y} &= [(1-U(t-t_c))T-D] \sin \Theta - mg \\ m\ddot{z} &= 0 \\ v^2 &= \dot{x}^2 + \dot{y}^2 \\ \text{Tan } \Theta &= \dot{y}/\dot{x} \end{aligned}\right\} \tag{5}$$

Alternatively,

T and t_c will be known, they will, in fact, depend somewhat upon environmental conditions during flight. The density lapse rate, h, may vary somewhat from standard. While wind components at the point of firing can be directly observed, the general or average components are not conveniently observable. Thus we need to estimate T, t_c, h, W_x, and W_z.

We shall first suppose that we have available noisy observations of u at certain times $t_1, t_2, ---, t_N$. We shall say that b_i is the observation at t_i, so that

$$b_i \cong u(t_i) \tag{13}$$

We propose to estimate T, t_c, h, W_x, and W_z in such a way as to minimize

$$S = \sum_{i=1}^{N} (u(t_i) - b_i)^T W_i (u(t_i) - b_i) \tag{14}$$

In equation (14) the matrices W_i are positive definite 4 by 4 weighting matrices whose nature will be discussed in a later section of this paper. A common form for the matrices W_i is shown below in equation (14-a).

$$W_i = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 000\lambda \end{bmatrix} \tag{14-a}$$

If the weighting matrices have this form, then the error functional S may be explicitly written as:

$$S = \sum_{i=1}^{N} [(\tilde{\dot{x}}_i - \dot{x}_i)^2 + (\tilde{\dot{y}}_i - \dot{y}_i)^2 + (\tilde{\dot{z}}_i - \dot{z}_i)^2 + (\tilde{y}_i - y_i)^2] \tag{14-b}$$

where λ is a positive dimensional constant which insures the homogeneity of the physical units in equation (14-b). It will be shown later that the value of λ is related to the square of the ratio of the velocity uncertainty to the position uncertainty.

Equations (20) and (21) are linear and thus easily integrable, given initial conditions. We have a complete set of initial conditions for (21),

$$\dot{Q}(0) = \emptyset$$

(where $\emptyset$ is the 4x5 null matrix). But, since u_o contains the unknown W_x and W_z, we do not have initial conditions for p. (The condition would be $p(0)=u_o$.)

We will, as usual, resort to the method of complementary functions in dealing with p. Let π be a particular solution to (20), obtained with any convenient initial conditions:

$$\dot{\pi} = f(u^o;c^o) + J(u^o;c^o)\cdot(\pi - u^o) - K(u^o;c^o)\cdot c^o, \quad \pi(0)=\pi_o \tag{25}$$

And let ϕ be a matrix of independent solutions to the homogeneous version of (20);

$$\dot{\phi} = J(u^o;c^o)\phi \quad , \; \phi(0) = \phi_o \tag{26}$$

where ϕ is a 4x4 matrix and ϕ_o is a constant 4x4 matrix with non-zero determinant. Then p can be represented as

$$p = \pi + \phi a \tag{27}$$

where a is a vector chosen to realize the initial conditions for p, that is such that

$$u_o = p(0) = \pi(0) + \phi(0)a \tag{28}$$

Now note that (from equations (7)):

$$u_o = \begin{bmatrix} V_g \cos \Theta_o + W_x \\ V_g \sin \Theta_o \\ W_z \\ 0 \end{bmatrix} \tag{29}$$

we obtain

$$\left.\begin{aligned}
\frac{\partial S_j^1}{\partial T} &= 2 \sum_{j=1}^{4} \sum_{i=1}^{N} \alpha_j(t_i) q_{j2} = 0 \\
\frac{\partial S_j^1}{\partial t_c} &= 2 \sum_{j=1}^{4} \sum_{i=1}^{N} \alpha_j(t_i) q_{j2} = 0 \\
\frac{\partial S_j^2}{\partial h} &= 2 \sum_{j=1}^{4} \sum_{i=1}^{N} \alpha_j(t_i) q_{j3} = 0 \\
\frac{\partial S_j^1}{\partial W_x} &= 2 \sum_{j=1}^{4} \sum_{i=1}^{N} \alpha_j(t_i) (\phi_{j1} + q_{j4}) = 0 \\
\frac{\partial S_j^2}{\partial W_z} &= 2 \sum_{j=1}^{4} \sum_{i=1}^{N} \alpha_j(t_i) (\phi_{j3} + q_{j5}) = 0
\end{aligned}\right\} \qquad (35)$$

where q_{jk} is the K^{th} element of Q_j and similarly for ϕ_{jk} and ϕ_j

Equations (35) provide a set of five linear algebraic equations in the five components of C^1. If well-conditioned, these may readily be solved for C^1. It is necessary first, however, to calculate J and K, since π, ϕ, and Q depend upon them. The columns of J are readily computed directly from (9):

From equations (1) and (3) we see

$$\frac{\partial D}{\partial h} = -yD \tag{41}$$

Thus

$$\frac{\partial f}{\partial h} = \begin{bmatrix} \dfrac{C_D \rho A v \dot{x}\, y}{2m} \\ \\ \dfrac{C_D \rho A\, v\, \dot{y}\, y}{2m} \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

Of course

$$\frac{\partial f}{\partial W_x} = \frac{\partial f}{\partial W_z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{43}$$

The columns of K are now given by (37), (39), (42), and (43).

With J and K at hand, π, ϕ, and Q can be obtained by numerical methods. C^1 can then be obtained from (35), and u^1 from (24). If we now set

$$\dot{u}^2 = f(u^1;c^1) + J(u^1;c^1)\cdot(u^2-u^1) + K(u^1;c^1)\cdot(c^2-c^1) \tag{44}$$

We can now proceed to obtain u^2 and c^2 just as we obtained u^1 and c^1. This process can be continued until either convergence or divergence becomes evident.

$$x = r\cos\phi\cos\beta \tag{45}$$

$$y = r\sin\phi \tag{46}$$

$$z = r\cos\phi\sin\beta \tag{47}$$

Differentiation of these quantities with respect to time permits us to express the projectile velocity vector in terms of the radar measured quantities.

$$u = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ y \end{bmatrix} = \begin{bmatrix} \dot{r}\cos\phi\cos\beta - r\dot{\phi}\sin\phi\cos\beta - r\dot{\beta}\cos\phi\sin\beta \\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{r}\cos\phi\sin\beta - r\dot{\phi}\sin\phi\sin\beta - r\dot{\beta}\cos\phi\cos\beta \\ r\sin\phi \end{bmatrix} \tag{48}$$

Returning for a moment to equations (8) and (13), let us write the components of the measured velocity vector b_i in the following component form:

$$b_i = \begin{bmatrix} \tilde{\dot{x}}_i \\ \tilde{\dot{y}}_i \\ \tilde{\dot{z}}_i \\ \tilde{y}_i \end{bmatrix} \tag{49}$$

In equation (49) the tilde indicates that the components of the observed vector b are not measured precisely but are contaminated by noise.

If we rewrite equation (14-b) we obtain:

$$S = \sum_{i=1}^{N} [(\tilde{\dot{x}}_i - \dot{x}_i)^2 + (\tilde{\dot{y}}_i - \dot{y}_i)^2 + (\tilde{\dot{z}}_i - \dot{z}_i)^2 + \lambda(\tilde{y}_i - y_i)^2] \tag{50}$$

Equation (50) states that our cost function S is the sum of the squares of the observed error residuals. It is now appropriate to make some comments about the constant λ appearing in equation (50).

Greater weight is now given in (52) to those observations in which we have the greatest faith. Hence, the quasilinearization process could be invoked toward the minimization of the quadratic function S* to obtain another estimate C* of the vector of unknown parameters. Because of a differential weighting of the residuals in S*, the estimate C* might be expected to have more desirable statistical properties than the estimate of C obtained by minimizing S.

In addition to the variances $\sigma^2_{\dot{x}_i}$, $\sigma^2_{\dot{y}_i}$, $\sigma^2_{\dot{z}_i}$ being unequal and possibly time varying, it is possible that at the same instant of time the errors in $\dot{x}_i, \tilde{\dot{y}}_i, \tilde{\dot{z}}_i, \tilde{y}_i$ might be cross-correlated. If this were the case and if the only error correlation was cross-correlation at the same instant of time, then in the case of multivariate gaussian observation errors, we might wish to proceed as follows. We let V_i denote the covariance matrix of the observation errors of the observations $\tilde{\dot{x}}_i$, $\tilde{\dot{y}}_i$, $\tilde{\dot{z}}_i$ made at time t_i. That is:

$$V_i = E[(b_i - u_i)(b_i - u_i)^T] \tag{53}$$

where E is the expected value operation and T is matrix transposition. Then if V_i could be computed a priori, or approximately inferred, we might wish to take advantage of our knowledge of observation error cross-correlation as well as variability of the observation variances.

Again, assuming quadrivariate gaussian observation error distributions, we are led to the following quadratic form:

$$S^{**} = \sum_{i=1}^{N} (b_i - u_i)^T V_i^{-1} (b_i - u_i) \tag{54}$$

Substituting equation (55) for the radar measured quantities into equation (48) we see that our measured velocity vector may be expressed as functions of the radar measurements as follows:

$$b = \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{z}} \\ y \end{bmatrix} = \begin{bmatrix} \dot{\tilde{r}}\cos\tilde{\phi}\cos\tilde{\beta} - \tilde{r}\dot{\tilde{\phi}}\sin\tilde{\phi}\cos\tilde{\beta} - \tilde{r}\dot{\tilde{\beta}}\cos\tilde{\phi}\sin\tilde{\beta} \\ \dot{\tilde{r}}\sin\tilde{\phi} + \tilde{r}\dot{\tilde{\phi}}\cos\tilde{\phi} \\ \dot{\tilde{r}}\cos\tilde{\phi}\sin\tilde{\beta} - \tilde{r}\dot{\tilde{\phi}}\sin\tilde{\phi}\sin\tilde{\beta} - \tilde{r}\dot{\tilde{\beta}}\cos\tilde{\phi}\cos\tilde{\beta} \\ \tilde{r}\sin\tilde{\phi} \end{bmatrix} \tag{56}$$

In equation (56) we have suppressed the time index i for the sake of notational brevity.

Now, even though the radar measured quantities are postulated to have known gaussian distributions, it is clear from equation (56) that the observed vector b is not distributed according to a trivariate gaussian law. If, however, the perturbing δ quantities have variances which are small compared to their true values, then it is possible to approximate the distribution of b by a trivariate gaussian distribution, or at least to approximately compute the variances and covariances or mixed moments of the quantities $\dot{\tilde{x}}$, $\dot{\tilde{y}}$, $\dot{\tilde{z}}$. It should be noted, incidentally, that although the errors in r, β, ϕ, etc. are assumed to be uncorrelated, the velocity component errors $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$ will be correlated.

If we let ζ_i denote the following 6x1 vector,

$$\zeta_i = \begin{bmatrix} r_i \\ \beta_i \\ \phi_i \\ \dot{r}_i \\ \dot{\beta}_i \\ \dot{\phi}_i \end{bmatrix} \tag{57}$$

the variances and covariances in the estimates of the velocity components cannot be precisely computed. We do, however, have the measured radar quantities $\tilde{\zeta}_i$ at time t_i so that we can evaluate the partial derivatives at the measured values $\tilde{r}_i$ --- etc., and by so doing obtain an estimate $\tilde{V}_i$ of the second order moments of the estimates of the target velocities. This estimate of the uncertainties can then be used to weight the residuals as in (52) according to:

$$\tilde{S}^* = \sum_{i=1}^{N} \left[\left(\frac{\dot{x}_i - \tilde{\dot{x}}_i}{\tilde{\sigma}_{\dot{x}_i}} \right)^2 + \left(\frac{x_i - \tilde{x}_i}{\tilde{\sigma}_{\dot{y}_i}} \right)^2 + \left(\frac{\dot{z}_i - \tilde{\dot{z}}_i}{\tilde{\sigma}_{\dot{z}_i}} \right)^2 + \left(\frac{y_i - \tilde{y}_i}{\tilde{\sigma}_{y_i}} \right)^2 \right] \tag{62}$$

where the estimated variances $\tilde{\sigma}^2_{\dot{x}_i}$, $\tilde{\sigma}^2_{\dot{y}_i}$, $\tilde{\sigma}^2_{\dot{z}_i}$ are the diagonal elements of $\tilde{V}_i$. This would allow the data to be weighted variably depending upon the relative geometry at the time of the ith observations. That is, even though the measured sensor vector $\tilde{\zeta}_i$ has a stationary error distribution, the variances in the computed estimates of the projectile velocity vector from the radar measurements, do have variances which are functions of geometry and time along the trajectory. Using $\tilde{S}^*$ as in (62) would allow one to account for the dependence of the residual uncertainty upon the projectile's actual position and velocity and to weight the more reliable data more heavily.

In a similar spirit, if one had confidence that valid information about cross-correlation of the velocity component errors was contained in the matrices $\tilde{V}_i$, then one might wish to work with the quadratic form

$$\tilde{S}^{**} = \sum_{i=1}^{N} (b_i - u_i)^T \tilde{V}_i^{-1} (b_i - u_i). \tag{63}$$

In equations (62) and (63) the estimates of the distributional second moments, which are used in the data weighting, are computable from the measured radar sensor data and an a priori knowledge of the radar sensor variances σ_r^2, --- σ_ϕ^2.

Now when ϵ has multivariate gaussian distribution then Q in (67) is proportional to the log likelihood function of the data and the estimate of a given by (68) is of course the maximum likelihood estimate. If ϵ is not normally distributed then $\hat{a}$ is still the minimum variance unbiased linear estimate even though the actual maximum likelihood estimate could conveivably be some nonlinear functional of the data which might have a smaller mean square error.

Although the analogy is a bit weak and far from compelling, we think that there may well be some merit in working with either of the cost functionals $\widetilde{S}^*$ or $\widetilde{S}^{**}$ as given in equations (62) and (63).

Unfortunately, the exigencies of our present work schedule, coupled with a lack of computer time and programming assistance, have precluded our performing the numerical experimentation which we had hoped to be able to do.